Hirn (G. A.) Exposition analytique et expérimentale de la Théorie Mécanique de la Chaleur. 3[e] Edition. Tome II. 8vo. *Paris* 1876. Sur l'Étude des moteurs thermiques et sur quelques points de la théorie de la Chaleur en général. 4to. 1876. The Author.

Jones (T. Wharton), F.R.S. Evolution of the Human Race from Apes and lower Animals a doctrine unsanctioned by Science. 8vo. *London* 1876. The Author.

Oil-Painting of the Hon. Robert Boyle, by Kneller, in gilt frame. Oil-Painting of Sir Charles Wheatstone, by Charles Martin, in gilt frame.

Thirteen Engraved Portraits of Francis Baily, Dr. Wm. Buckland, Sir Henry De la Beche, Faraday, Sir John Franklin, Thomas Graham, Sir Henry Holland, Moll, Sir Roderick Murchison, Sir James Ross, Canon Sedgwick (framed and glazed).

Six Photographic Portraits of Amici, Liebig, Priestley, &c. (framed and glazed).

The Bequest of the late Sir Charles Wheatstone, F.R.S.

February 3, 1876.

WILLIAM SPOTTISWOODE, M.A., Treasurer and Vice-President, in the Chair.

The Presents received were laid on the table, and thanks ordered for them.

The following Papers were read:—

I. "On Formulæ of Verification in the Partition of Numbers." By J. W. L. Glaisher, M.A., F.R.S. Received December 18, 1875.

In writing down all the partitions of a number it is convenient to have tests which will verify that none have been omitted; and it is the main object of this paper to give formulæ which are available for this purpose.

At the Edinburgh Meeting of the British Association (Report 1871, Transactions of the Sections, p. 23), Sylvester communicated the elegant and useful test-formula

$$\text{I.} \qquad \Sigma(1-x+xy-xyz+\text{ \&c.})=0,$$

where in any partition x denotes the number of 1's present, y the number

of 2's, z the number of 3's, and the Σ extends to all the partitions of the given number n; so that $\Sigma 1 = \mathrm{N}$, the total number of the partitions of n. A single formula of verification, however, is rarely sufficient to afford a perfect test of accuracy, as the effects produced by certain omissions may cancel one another. In the present case, for example, as the terms are alternately positive and negative, if the omitted partition contained one 1 and no 2 it would appear as 1 in the first term and as 1 in the second term, and its omission would thus not be pointed out. Having had occasion several times to employ Sylvester's equation I., and having found the need of some additional formula to be used in connexion with it, I was led to seek for identities which would afford similar verifications. It seemed natural first to investigate what I. became when all the signs were positive.

Starting from the identity

$$1+\frac{t}{1-t}+\frac{t^3}{1-t\,.\,1-t^2}+\frac{t^6}{1-t\,.\,1-t^2\,.\,1-t^3}+\&\mathrm{c.}=1+t\,.\,1+t^2\,.\,1+t^3\ldots,$$

and dividing throughout by $1-t\,.\,1-t^2\,.\,1-t^3\ldots$ we have

$$\frac{1}{1-t\,.\,1-t^2\,.\,1-t^3\ldots}+\frac{t}{(1-t)^2\,1-t^2\,.\,1-t^3\ldots}+\frac{t^3}{(1-t)^2(1-t^2)^2\,1-t^3\ldots}$$
$$+\&\mathrm{c.}$$
$$=\frac{1+t\,.\,1+t^2\,.\,1+t^3\ldots}{1-t\,.\,1-t^2\,.\,1-t^3\ldots}$$
$$=(1+2t+2t^2+\&\mathrm{c.})\,(1+2t^2+2t^4+\&\mathrm{c.})\,(1+2t^3+2t^6+\&\mathrm{c.})\ldots;$$

whence, equating the coefficients of t^n,

$$\text{II.}\qquad \Sigma(1+x+xy+xyz+\&\mathrm{c.})=\Sigma 2^r,$$

where r is the number of different elements contained in a partition. Take as an example $n=9$; the partitions are :—

9	8+1	7+1+1	6+1+1+1	5+1+1+1+1
	7+2	6+2+1	5+2+1+1	4+2+1+1+1
	6+3	5+3+1	4+3+1+1	3+3+1+1+1
	5+4	5+2+2	4+2+2+1	3+2+2+1+1
		4+4+1	3+3+2+1	2+2+2+2+1
		4+3+2	3+2+2+2	
		3+3+3		

4+1+1+1+1+1	3+1+1+1+1+1+1
3+2+1+1+1+1	2+2+1+1+1+1+1
2+2+2+1+1+1	2+1+1+1+1+1+1+1

1+1+1+1+1+1+1+1+1

Here N (the number of partitions) $=30$, Σx (the number of 1's) $=67$, $\Sigma xy=47$, and $\Sigma xyz=10$. Also the number of partitions involving only

one element $=3$, the number involving two elements $=17$, and the number involving three $=10$; and I. and II. become

$$30-67+47-10=0,$$
$$30+67+47+10=3\,.\,2+17\,.\,2^2+10\,.\,2^3.$$

Combining the formulæ I. and II., we have

$$\Sigma(1+xy+xyzw+\&c.)=\Sigma(x+xyz+\&c.)=\tfrac{1}{2}\Sigma 2^r=\Sigma 2^{r-1},$$

where $\Sigma 2^{r-1}$ may be written $\Sigma 2$, s denoting the number of *changes* in a partition. The two formulæ taken together, as just written, form of course a much better verification than either singly.

There are two subsidiary verifications connected with I. and II. which are also worth attention. From Jacobi's equation ('Fundamenta Nova,' p. 185),

$$\frac{1-t\,.\,1-t^2\,.\,1-t^3\ldots}{1+t\,.\,1+t^2\,.\,1+t^3\ldots}=1-2t+2t^4-2t^9+2t^{16}-\&c.,$$

we see that

III. $$\Sigma\pm 2^{r-1}=1,\ -1, \text{ or } 0,$$

according as n is an even square, an uneven square, or not a square. The sign $+$ is to be taken if the partition contains an even number of terms, and the sign $-$ if the number is uneven.

Also from the same identity inverted, viz. from

$$\frac{1+t\,.\,1+t^2\,.\,1+t^3\ldots}{1-t\,.\,1-t^2\,.\,1-t^3\ldots}=\frac{1}{1-2t+2t^4-2t^9+2t^{16}-\&c.},$$

we see that

IV. $$\Sigma 2^r=(-)^n(\mathrm{R}-\mathrm{R}'),$$

where $\mathrm{R}=$the number of representations of n as the sum of an even number of squares, and R' the number of representations as an uneven number of squares.

In the case of $n=9$, of the three partitions involving only one element, all consist of an uneven number of terms; of the seventeen partitions involving two elements, in nine the number of terms is even, and in eight uneven; and of the ten partitions involving three elements, in five the number is even and in five uneven; so that since 9 is an uneven square, III. gives

$$\tfrac{1}{2}(-3\,.\,2+9\,.\,2^2-8\,.\,2^2+5\,.\,2^3-5\,.\,2^3)=-1.$$

The partitions of 9 into squares are four in number, viz. 9, $4+4+1$, $4+1+1+1+1+1$, $1+1+1+1+1+1+1+1+1$; and these give rise respectively to $2, 3\,.\,2^3, 6\,.\,2^6, 2^9$ representations; so that substituting in IV.

$$30+67+47+10=(-)^9\,(384-2-24-512).$$

The following are also formulæ of verification :—

V. $$\Sigma(x-2xy+3xyz-\&c.)=\psi n,$$

where ψn denotes, as also in VI. and VII., the number of divisors of n, unity and n included. This formula does not verify the value of N, but it verifies Σx, Σxy, &c.; for $n=9$, $\psi n=3$; and it gives

$$67-2.47+3.10=3.$$

VI. $$\Sigma(1-2y+3yz-4yzw+\&c.)=\psi(n+1)-\psi n.$$

For $n=9$ (Σy, the total number of 2's) $=26$, $\Sigma yz=9$, $\Sigma yzw=1$, and

$$30-2.26+3.9-4.1=4-3.$$

VII. $$\Sigma\big(1-2[y]_1+3[yz]_2-4[yzw]_3+\&c.\big)=\psi(n+2)-1,$$

where $[\ ]_r$ denotes that the enclosed quantity is zero for every partition which does not contain at least r 1's.

For $n=9$, $\Sigma[y]_1=19$, $\Sigma[yz]_2=3$, and

$$30-2.19+3.3=2-1.$$

VIII. $$N-1=\Sigma(x-1-x-2.y+x-3.yz-\&c.),$$

where, as throughout, the point is written in place of brackets (ex. gr. $x-2.y$ stands for $(x-2)y$), and a negative factor is to be treated as zero.

For $n=9$, $\Sigma(x-1)=45$, $\Sigma(x-2.y)=17$, $\Sigma(x-3.yz)=1$, and

$$30-1=45-17+1.$$

IX. $$N-n=\Sigma(\{x-3\}_2+\{x-5\}_3+\{x-7\}_4+\&c.),$$

where $\{\ \}_r$ denotes that the enclosed quantity is zero for every partition in which an element $>r$ appears.

For $n=9$, $\Sigma\{x-3\}_2=2+4+6=12$, $\Sigma\{x-5\}_3=1+2+4=7$, $\Sigma\{x-7\}_4=2$; and the formula gives

$$30-9=12+7+2.$$

X. $$N-n=\Sigma\big(\{x-1.y\}_2+\{x-3.yz\}_3+\{x-6.yzw\}_4+\&c.,$$

where $\{\ \}_r$ has the same meaning as in IX., and 1, 3, 6 .... are the triangular numbers.

For $n=9$, the formula gives

$$30-9=20+1.$$

XI. If n be uneven, $\Sigma\big(1-[x+1]^1-[x+1.y+1.z+1]^3-\&c.\big)=0$;

and if n be even, $\Sigma\big(1-[1]^0-[x+1.y+1]^2-[x+1\ldots w+1]^4-\&c.\big)=0$,

where $x=$ the number of 2's, $y=$ the number of 4's &c. in any partition, and $[\ ]^r$ denotes that the enclosed quantity is equal to zero unless the partition contains exactly r ones, and no other uneven element. Thus $\Sigma[1]^0$ is equal to the number of partitions formed wholly of even elements. (x, y, z ... may, of course, have zero values.)

For $n=9$, the first formula gives

$$30-12-10-5-2-1=0.$$

XII. $$\Sigma_1 2^{r}=\nu+P_2^1+P_{2,4}^{1,3}+P_{2,4,6}^{1,3,5}+\&c.,$$

where Σ_1 refers only to partitions in which all the elements are uneven, ν denotes the number of partitions of n into uneven elements (so that $\Sigma_1 1=\nu$); and, for example, $P_{2,4,6}^{1,3,5}$ denotes the number of partitions of n into the elements 2, 4, 6, 1, 3, 5, 7 .... (2, 4, 6 being the only admissible even elements, but all uneven elements being admissible), and in which 1, 3, 5 must each occur at least once; so that in the partitions all the uneven numbers and the (even) suffixes may appear, and the (uneven) exponents *must* all appear. (Thus we might have written $P_{2,4,6}^9$, &c. for $P_{2,4,6}^{1,3,5}$, &c.)

For $n=9$, we have

$$30=8+14+7+1.$$

The notation being as in XII.,

XIII. $$0=\nu-P_2^1+P_{2,4}^{1,3}-P_{2,4,6}^{1,3,5}+\&c.$$

For $n=9$, $$0=8-14+7-1.$$

Combining XII. with XIII., we have

$$\nu+P_{2,4}^{1,3}+P_{2,4,6,8}^{1,3,5,7}+\&c.=P_2^1+P_{2,4,6}^{1,3,5}+\&c.=\tfrac{1}{2}\Sigma 2^{r}.$$

XIV. $$P-P^2+P^{2,4}-P^{2,4,6}+\&c.=1 \text{ or } 0,$$

according as n is or is not a triangular number.

Here the notation is practically the same as in XII. and XIII.; P denotes the number of partitions into uneven elements, P^2 the number of partitions into the elements 2, 1, 3, 5 . . ., in which 2 must appear, $P^{2,4}$ into the elements 2, 4, 1, 3, 5 . . ., in which 2 and 4 must both appear, and so on.

For $n=9$, the formula gives

$$8-11+3=0.$$

The formulæ V. to XIV., which have just been written down, are merely translations into analytical language of the identities :—

(V.) $$\frac{t}{1-t}+\frac{t^2}{1-t^2}+\frac{t^3}{1-t^3}+\&c.=\frac{t}{(1-t)^2\,.\,1-t^2\,.\,1-t^3\ldots}$$
$$-\frac{2t^3}{(1-t)^2(1-t^2)^2\,.\,1-t^3\ldots}+\frac{3t^6}{(1-t)^2(1-t^2)^2(1-t^3)^2\ldots}-\&c.$$

(VI.) $$\left(\frac{1}{t}-1\right)\left(\frac{t}{1-t}+\frac{t^2}{1-t^2}+\&c.\right)=\frac{1}{1-t\,.\,1-t^2\,.\,1-t^3\ldots}$$
$$-\frac{2t^2}{1-t\,.\,(1-t^2)^2\,.\,1-t^3}+\&c.$$

$$\text{(VII.)}\ \frac{1}{1-t^2}+\frac{t}{1-t^3}+\frac{t^2}{1-t^4}+\&c.=\frac{1}{1-t.1-t^2.1-t^3\ldots}$$
$$-\frac{2t^3}{1-t.(1-t^2)^2.1-t^3\ldots}+\frac{3t^6.t}{1-t.(1-t^2)^2(1-t^3)^2\ldots}-\&c.$$

$$\text{(VIII.)}\ \frac{1}{1-t}=\frac{1}{1-t.1-t^2.1-t^3\ldots}-\frac{t^2}{(1-t)^2.1-t^2.1-t^3\ldots}$$
$$+\frac{t^5}{(1-t)^2(1-t^2)^2 1-t^3\ldots}-\&c.$$

$$\text{(IX.)}\ \frac{1}{1-t.1-t^2.1-t^3\ldots}=\frac{1}{1-t}+\frac{t^2}{(1-t)^2}+\frac{t^4}{(1-t)^2.1-t^2}$$
$$+\frac{t^6}{(1-t)^2.1-t^2.1-t^3}+\&c.$$

$$\text{(X.)}\ \frac{1}{1-t.1-t^2.1-t^3\ldots}=1+\frac{t}{(1-t)^2}+\frac{t^4}{(1-t)^2(1-t^2)^2}$$
$$+\frac{t^9}{(1-t)^2(1-t^2)^2(1-t^3)^2}+\&c.$$

$$\text{(XI.)}\ \frac{1}{1-t.1-t^2.1-t^3\ldots}=\frac{1}{1-t^2.1-t^4.1-t^6\ldots}$$
$$+\frac{t}{(1-t^2)^2.1-t^4.1-t^6\ldots}+\frac{t^2}{(1-t^2)^2(1-t^4)^2.1-t^6\ldots}+\&c.$$

$$\text{(XII.)}\ \frac{1+t.1+t^3.1+t^5\ldots}{1-t.1-t^3.1-t^5\ldots}=\frac{1}{1-t.1-t^3.1-t^5\ldots}$$
$$+\frac{t}{(1-t^2).1-t.1-t^3\ldots}+\frac{t^4}{(1-t^2)(1-t^4).1-t\ldots}+\&c.$$

$$\text{(XIII.)}\quad 1 =\frac{1}{1-t.1-t^3.1-t^5\ldots}-\frac{t}{(1-t^2).1-t.1-t^3\ldots}$$
$$+\frac{t^4}{(1-t^2)(1-t^4).1-t\ldots}-\&c.$$

$$\text{(XIV.)}\ 1+t+t^3+t^6+t^{10}+\&c.=\frac{1}{1-t.1-t^3.1-t^5\ldots}$$
$$-\frac{t^2}{(1-t^2).1-t.1-t^3\ldots}+\frac{t^6}{(1-t^2)(1-t^4).1-t\ldots}-\&c.;$$

all of which can be deduced without difficulty from the identities,

$$(\alpha)\ 1+tu.1+t^2u.1+t^3u\ldots=1+\frac{tu}{1-t}+\frac{t^3u^2}{1-t.1-t^2}$$
$$+\frac{t^6u^3}{1-t.1-t^2.1-t^3}+\&c.$$

$$(\beta)\ \frac{1}{1-tu.1-t^2u.1-t^3u\ldots}=1+\frac{tu}{1-t}+\frac{t^2u^2}{1-t.1-t^2}$$
$$+\frac{t^3u^3}{1-t.1-t^2.1-t^3}+\&c.$$

$$(\gamma)\ \frac{1}{1-tu\,.\,1-t^2u\,.\,1-t^3u\,\ldots}=1+\frac{t}{1-t}\cdot\frac{u}{1-tu}+\frac{t^4}{1-t\,.\,1-t^2}\cdot$$

$$\frac{u^2}{1-tu\,.\,1-t^2u}+\frac{t^9}{1-t\,.\,1-t^2\,.\,1-t^3}\cdot\frac{u^3}{1-tu\,.\,1-t^2u\,.\,1-t^3u}+\&c.$$

$$(\delta)\ \frac{1-t^2\,.\,1-t^4\,.\,1-t^6\,\ldots}{1-t\,.\,1-t^3\,.\,1-t^5\,\ldots}=1+t+t^3+t^6+t^{10}+\&c.\,;$$

of which (α) and (β) are Euler's well known formulæ, and (γ) and (δ) are given by Jacobi ('Fundamenta Nova,' pp. 180, 185).

Of the fourteen verification formulæ the first two are generally most convenient; and, taken in connexion with III. and IV., the four equations form an interesting system of mutually related verifications. All the formulæ can be used with great facility after a little practice, but some are evidently far preferable to others. Nos. II., III. and IV. were communicated to the Bristol Meeting of the British Association (1875).

The following formulæ, XV. to XX., involve the consideration of the partitions of a number in which repetitions are excluded. If by $Q\,(a, b, c\ldots)\,n$ be denoted the number of partitions without repetitions of n into the elements $a, b, c\ldots$, and by $P\,(a, b, c\ldots)\,n$ the number of partitions into the same elements with repetitions, then, from the identity

$$2\,.\,1+t\,.\,1+t^2\,.\,1+t^3\ldots=1+\frac{1}{1-t}+\frac{t}{1-t\,.\,1-t^2}+\frac{t^3}{1-t\,.\,1-t^2\,.\,1-t^3}+\&c.,$$

which is derivable at once from (α), we have

$$\text{XV.}\quad 2Q(1,2,3\ldots)n=1+P^{1}(1,2)n+P^{1,2}(1,2,3)n+P^{1,2,3}(1,2,3,4)n+\&c.,$$

where $P^{1,2\ldots r}(1,2\ldots r+1)n$ denotes the number of partitions of n into the elements $1, 2\ldots r+1$, in which all except the highest must appear at least once.

For $n=9$, $Q(1,2,3\ldots)n=8$, and the formula gives

$$2\,.\,8=1+5+7+3.$$

From the identity

$$\frac{1}{1+t\,.\,1+t^2\,.\,1+t^3\ldots}=1-t\,.\,1-t^3\,.\,1-t^5\ldots,$$

we have

$$\text{XVI.}\qquad P^{\text{even}}n-P^{\text{uneven}}n=(-)^nQ(1,3,5\ldots)n,$$

where $P^{\text{even}}n$ denotes the number of partitions of n that contain an even number of terms, and $P^{\text{uneven}}n$ the number that contain an uneven number. If n be even, all the partitions in $Q(1,3,5\ldots)n$ are included in $P^{\text{even}}n$; and if n be uneven, all are included in $P^{\text{uneven}}n$; so that we have the theorem that if all the partitions of n into uneven elements without repetitions be left out of consideration, then of the rest the number of partitions in which the number of terms is even is equal to the number

of partitions in which the number of terms is uneven. For the case $n=9$, the partitions which contain only uneven elements without repetitions are two in number, viz. 9 and $1+3+5$; and of the rest, fourteen consist of an even, and fourteen of an uneven number of terms.

We have, as above,

$$(\varepsilon)\ \frac{1}{1-t\,.\,1-t^3\,.\,1-t^5\ldots}=1+t\,.\,1+t^2\,.\,1+t^3\ldots$$
$$=1+t+t^2(1+t)+t^3(1+t)(1+t^2)$$
$$+t^4(1+t)(1+t^2)(1+t^3)+\&c.;$$

whence

$$\frac{1}{1-t^3\,.\,1-t^5\,.\,1-t^7\ldots}=1+t^{2+1}+t^{3+2}(1+t)+t^{4+3}(1+t)(1+t^2)+\&c.,$$

and we obtain

XVII. $P(3,5,7\ldots)n=$number of partitions without repetitions into the elements $1, 2, 3\ldots$, in which the two greatest parts are consecutive (*i. e.* differ by unity).

And also, by multiplying (ε) by $1-t$, we find that each side of XVII. $=Q(3, 5, 6, 7, 9\ldots)n$; so that if all the partitions without repetitions of a number n be written down, then

XVIII. the number of partitions not involving $1, 2, 4, 8, 16\ldots =$ the number of partitions in which the two greatest parts are consecutive.

In the case of $n=9$, XVII. gives $2=2$; and XVIII. of course also gives $2=2$, there being only two partitions without repetitions in which $1, 2\ldots$ are not involved, viz. 9 and $6+3$, and two in which the two greatest parts are consecutive, viz. $4+3+2$ and $5+4$.

Euler's identity,

$$1-t\,.\,1-t^2\,.\,1-t^3\ldots=1-t-t^2+t^5+t^7-t^{12}-\&c.$$

(where the exponents are the pentagonal numbers), gives the theorem that

$$\text{XIX.}\qquad Q^{\text{even}}n-Q^{\text{uneven}}n=(-1)^m \text{ or } 0,$$

according as n is or is not of the form $\frac{1}{2}(3m^2\pm m)$. Thus we see that, considering only partitions without repetitions, the number of the partitions of a non-pentagonal number into an even number of parts is always equal to the number of its partitions into an uneven number; and that if the number be pentagonal the numbers of even and uneven partitionments differ only by unity. If $n=9$, $Q^{\text{even}}n=Q^{\text{uneven}}n=4$.

We deduce from (δ) that

$$1+t\,.\,1+t^3\,.\,1+t^5\ldots\{1-t^4\,.\,1-t^8\,.\,1-t^{12}\ldots\}=1+t+t^3+t^6+t^{10}+\&c.;$$

and by Euler's identity, just quoted, the quantity in $\{\ \}$

$$=1-t^4-t^8+t^{20}+t^{28}-t^{48}-\&c.;$$

whence

$$\text{XX.}\quad Q(1,3,5\ldots)n-Q(1,3,5\ldots)(n-4)-Q(1,3,5\ldots)(n-8)$$
$$+Q(1,3,5\ldots)(n-20)+\&c.=1 \text{ or } 0,$$

according as n is or is not a triangular number. The general term on the left-hand side is $(-)^m Q(1, 3, 5 \ldots)(n-6m^2 \pm 2m)$. For $n=9$, the formula gives

$$2-1-1=0.$$

There is a proposition of Euler's which follows immediately from (ϵ), viz. that the number of partitions of a number into the uneven elements 1, 3, 5 ... is equal to the number of partitions without repetitions into the elements 1, 2, 3 ..., or, in the notation already used, that $P(1, 3, 5 \ldots)n = Q(1, 2, 3 \ldots)n$. It is natural to suppose that this striking theorem should admit of a simple demonstration by converting the partitions of the one system into those of the other according to some rule, such as in Mr. Ferrers's method of showing the identity of the numbers of partitionments into the elements $1, 2, 3 \ldots m$, and into parts not exceeding m in number (Phil. Mag. S. 4. vol. v. p. 201, 1853). The conversion, though not so simple as that just referred to, is nevertheless elegant and elementary; it in effect consists of a transformation into the binary scale. Thus consider a partition into uneven numbers containing α 1's, β 3's, γ 5's, &c. This is transformable at once into $1 . \alpha+3 . \beta+5 . \gamma$ +&c. Now express $\alpha, \beta, \gamma \ldots$ in the binary scale so that $\alpha=2^a+2^{a'}+$ &c., $\beta=2^b+2^{b'}+$ &c., $\gamma=2^c+2^{c'}+$ &c., then the partition becomes $1(2^a+$ &c.$)+3(2^b+$ &c.$)+5(2^c+$ &c.$)+$ &c., viz. becomes $2^a+2^{a'}+$&c. $+3 . 2^b+3 . 2^{b'}+$&c.$+5 . 2^c+5 . 2^{c'}+$&c., in which no two parts are identical, since a number can be expressed in only one way in the form $2^m A$, A being uneven.

To illustrate this take Euler's example of $n=10$; the partitions into uneven elements are

(i) 9+1,	(vi) 5+3+1+1,
(ii) 7+3,	(vii) 3+3+3+1,
(iii) 7+1+1+1,	(viii) 3+3+1+1+1+1,
(iv) 5+5,	(ix) 3+1+1+1+1+1+1+1,
(v) 5+1+1+1+1+1,	(x) 1+1+1+1+1+1+1+1+1+1.

By the conversion (i) and (ii) remain unaltered; (iii) $=7+1.3=7+1(2+1)=7+2+1$; (iv) $=5 . 2=10$; (v) $=5+1 . 5=5+1(2^2+1)=5+4+1$; (vi) $=5+3+1 . 2=5+3+2$; (vii) $=3 . 3+1=3(2+1)+1=6+3+1$; (viii) $=3 . 2+1 . 4=6+4$; (ix) $=3+1 . 7=3+1(2^2+2+1)=3+4+2+1$; (x) $=1 . 10=1(2^3+2)=8+2$; thus giving the partitions without repetitions, viz.

9+1,	5+3+2,
7+3,	6+3+1,
7+2+1,	6+4,
10,	3+4+2+1,
5+4+1,	8+2.

To reverse the process it is only necessary to remark that every number is either uneven, or twice an uneven number, or four times an uneven number, or eight times an uneven number, &c., so that every sum of different elements can be uniquely reduced to the form $1+3 \, . \, \alpha+5 \, . \, \beta+$&c. (The practical reduction is very easy, *ex. gr.* $14+12+5+3+2$ converts into $7+7+3+3+3+3+5+3+1+1$.) A little consideration shows that no two different partitions of one system can convert into the same partition of the other, so that every partition of either system corresponds uniquely to a partition of the other; and the numbers of partitions in the two systems are thus equal. Several conversions of $P(1, 3, 5 \ldots)n$ are given in the Phil. Mag. for April 1875 (S. 4. vol. xlix. pp. 307–311); but when writing that note I failed to obtain the connexion between the partitions into 1, 3, 5 ... with repetitions and the partitions into 1, 2, 3 ... without repetitions.

II. "On the Development and Succession of the Poison-fangs of Snakes." By Charles S. Tomes, M.A. Communicated by John Tomes, F.R.S. Received December 28, 1875.

(Abstract.)

At the conclusion of a paper upon the development of the teeth of Ophidia, published in the first part of the Philosophical Transactions for 1875, I noted that there were peculiarities, which I had not then been able to understand, in the succession and the development of the poison-fangs. Having reviewed the literature of the subject in that and in a preceding paper on the development of Amphibian teeth, I will pass at once to the description of the special features which distinguish the development of poison-fangs. Poisonous snakes are divided into two groups—those which have a shortened movable maxillary bone, which carries the poison-fang and another tooth; and those which have the maxillary bone longer, immovable, and often carrying other teeth behind the poison-fang.

In the former, or viperine poisonous snakes, the poison-fang is very long, and, when out of use, lies recumbent; in the latter, or colubrine poisonous snakes, it is, from the maxillary bone being fixed, constantly erect (Günther's 'Reptiles of British India,' p. 165).

As fresh specimens are indispensable for a complete investigation of developmental peculiarities, I have only been able to examine one of the colubrine group, viz. the Indian cobra.

Of it one may say, roughly speaking, that the poison-fangs are developed just like any other Ophidian teeth, for a description of which I must refer to my former paper, save only that the tooth-germs are necessarily individually modified so as to produce the characteristic canaliculated poison-tooth.